Arrays

2.2 The Array as an Abstract Data Type

Array:

■ **A set of pairs: <index, value> (correspondence or mapping)**

■ **Two operations: retrieve, store**

Now we will use the C++ class to define an ADT.

GeneralArray

class GeneralArray {

// a set of pairs <index, value> where for each value of

// index in IndexSet there is a value of type **float**. IndexSet is

// a finite ordered set of one or more dimensions.

public:

GeneralArray(**int** j, RangeList list, **float** initValue = defaultValue)**;**

// The constructor GeneralArray creates a **j**

// dimensional array of floats; the range of the kth

// dimension is given by the kth element of list.

// For all i∈IndexSet, insert <i, initValue> into the array.

float Retrieve(index i)**;**
// **if** (i∈IndexSet) **return** the float associated with i in the
// array;**else** throw an exception.

void Store(index i, **float** x)**;**
// **if** (i∈IndexSet) replace the old value associated with i
// by x**;** **else** throw an exception.
}; //end of GeneralArray

Note:

- **Not necessarily implemented using consecutive memory**
- **Index can be coded any way**
- **GeneralArray is more general than C++ array as it is more flexible about the composition of the index set**
- **To be simple, we will hereafter use the C++ array**

Array can be used to implement other abstract data types. The simplest one might be:

Ordered or linear list.

Example:

(Sun, Mon, Tue, Wed, Thu, Fri, Sat)

(2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K, A)

( )   // empty list

More generally, An ordered list is either empty or $(a_0, a_1, .., a_{n-1})$.   // index important

Main operations:

(1) Find the length, n, of the list.

(2) Read the list from left to right (or right to left)

(3) Retrieve the ith element, $0 \le i < n$.

(4) Store a new value into the ith position, $0 \le i < n$.

(5) Insert a new element at position i, $0 \le i < n$, causing elements numbered i, i+1,…n-1 to become numbered i+1, i+2,…n.

(6) Delete the element at position i, $0 \le i < n$, causing elements numbered i+1, i+2,…n-1 to become numbered i, i+1,…n-2.

How to represent ordered list efficiently?

- **Sequential mapping**
  - **Use array: a_i ←→ index i**
- **Complexity**
  - **Random access any element in**
    - **O(1).**
  - **Operations (5) and (6)?**
    - **Data movement**
    - **O(n)**

Now let us look at a problem requiring ordered list.

Problem:

Build an ADT for the representation and manipulation of symbolic polynomials.

$A(x)=3x^2+ 2x+4$

$B(x)=x^4+ 10x^3+ 3x^2+1$

Degree: the largest exponent

ADT Polynomial

class Polynomial {

// $p(x)=a_0x^{e0}+,\ldots,+ a_nx^{en}$; a set of ordered pairs of $<e_i, a_i>$,

// where a_i is a nonzero **float** coefficient and e_i is a

// non-negative exponent

public:

Polynomial ( )**;**

// Construct the polynomial $p(x)=0$

void AddTerm (Exponent e, Coefficient c)**;**

// add the term <e,c> to ***this**, so that it can be initialized

Polynomial Add (Polynomial poly)**;**

// return the sum of the polynomials ***this** and poly

Polynomial Mult (Polynomial poly)**;**

// return the product of the polynomials ***this** and poly

float Eval (**float** f)**;**

// evaluate polynomial ***this** at f and return the result

}

Polynomial Representation

Let a be $A(x)=a_n x^n + a_{n-1} x^{n-1} +,\ldots,+ a_1 x + a_0$

Representation 1

private:

int degree; // degree ≤ MaxDegree

float coef[MaxDegree+1];

a.degree= ?

n;

MaxDegree?

a.coef[i] = ?

a_{n-i}, 0 ≤i≤ n

Simple algorithms for many operations.

Representation 2

When a.degree << MaxDegree, representation 1 is very poor in memory use. To improve, define variable sized data member as:

```
private:
   int degree;
   float *coef;
```

```
Polynomial::Polynomial(int d)
 {
   int degree=d;
   coef= new float[degree+1];
 }
```

Representation 2 is still not desirable.

For instance, $x^{1000}+1$

makes 999 entries of the coef be zero.

So, we store only the none zero terms:

Representation 3

$$A(x) = b_m x^{em} + b_{m-1} x^{em-1} +,\ldots,+ b_0 x^{e0}$$

Where $b_i \neq 0$, $e_m > e_{m-1} >,\ldots, e_0 \geq 0$

```
class Polynomial; // forward declaration
class Term {
friend Polynomial;
private:
    float coef; // coefficient
    int exp;     // exponent
};

class Polynomial {
public:
   ……
private:
   Term *termArray;
   int capacity; // size of termArray
   int terms; // number of nonzero terms
}
```

For $A(x) = 2x^{1000} + 1$

A.termArray looks like:

coef	2	1		
exp	1000	0		

Many zero --- good

Few zero --- ?

not very good

may use twice as much space as in presentation 2.

Polynomial Addition

Use presentation 3 to obtain C = A +B.

Idea:

Because the exponents are in descending order, we can adds A(x) and B(x) term by term to produce C(x).

The terms of C are entered into its termArray by calling function NewTerm.

If the space in termArray is not enough, its capacity is doubled.

```
Polynomial  Polynomial::Add (Polynomial b)
{   // return the sum of the polynomials *this and b.
  Polynomial c;
  int aPos=0, bPos=0;
  while (( aPos < terms) && (b < b.terms))
     if (termArray[aPos].exp==b.termArray[bPos].exp) {
                 float t = termArray[aPos].coef + termArray[bPos].coef
     if ( t )
          c.NewTerm (t, termArray[aPos].exp);
          aPos++;  bPos++;
     }
     else if (termArray[aPos].exp < b.termArray[bPos].exp) {
            c.NewTerm (b.termArray[bPos].coef,
                           b.termArray[bPos].exp);
       bPos++;
     }
```

```
   else {
      c.NewTerm (termArray[aPos].coef, termArray[aPos].exp);
       aPos++;
    }
 // add in the remaining terms of *this
for ( ; aPos < terms; aPos++ )
    c.NewTerm(termArray[aPos].coef, termArray[aPos].exp );
// add in the remaining terms of b
for ( ; bPos < b.terms; bPos++ )
c.NewTerm(b.termArray[bPos].coef, b.termArray[bPos].exp);
return c;
}
```

```
void Polynomial::NewTerm(const float theCoeff,
                         const int theExp)
{ // add a new term to the end of termArray.
  if (terms == capacity)
  { // double capacity of termArray
     capacity *= 2;
     term *temp = new term[capacity]; // new array
     copy(termArray, termAarry + terms, temp);
     delete [ ] termArray;  // deallocate old memory
     termArray = temp;
  }
 termArray[terms].coef = theCoeff;
 termArray[terms++].exp = theExp;
}
```

Analysis of Add:

Let m, n be the number of nonzero terms in a and b respectively.

- **line 3 and 4---O(1)**
- **in each iteration of the while loop, aPos or bPos or both increase by 1, the number of iterations of this loop ≤ m+n-1**
- **if ignore the time for doubling the capacity, each iteration takes O(1)**
- **line 20--- O(m), line 23--- O(n)**

Asymptotic computing time for Add: O(m+n)

Analysis of doubling capacity:

- **the time for doubling is linear in the size of new array**
- **initially, c.capacity is 1**
- **suppose when Add terminates, c.capacity is 2^k**
- **the total time spent over all array doubling is**

$$O\left(\sum_{i=1}^{k} 2^{i}\right) = O(2^{k+1}) = O(2^{k})$$

- **since c.terms > 2^{k-1} and m + n ≥ c.terms, the total time for array doubling is**

$$O(c.terms) = O(m + n)$$

- **so, even consider array doubling, the total run time of Add is O(m + n).**
- **experiments show that array doubling is responsible for very small fraction of the total run time of Add.**

Exercises: P93-2,6, P94-9

Sparse Matrices

Introduction

A general matrix consists of m rows and n columns (m × n)of numbers, as:

	0	1	2
0	-27	3	4
1	6	82	-2
2	109	-64	11
3	12	8	9
4	48	27	47

Fig.2.2(a) 5×3

	0	1	2	3	4	5
0	15	0	0	22	0	-15
1	0	11	3	0	0	0
2	0	0	0	-6	0	0
3	0	0	0	0	0	0
4	91	0	0	0	0	0
5	0	0	28	0	0	0

Fig. 2.2(b) 6×6

A matrix of m × m is called a square.

A matrix with many zero entries is called sparse.

Representation:

- **A natural way ---**
  - **a[m][n]**
  - **access element by a[i][j], easy operations. But**
  - **for sparse matrix, wasteful of both memory and time.**
- **Alternative way ---**
  - **store nonzero elements explicitly. 0 as default.**

SparseMatrix

```
class SparseMatrix
 { // a set of <row, column, value>, where row, column are
  // non-negative integers and form a unique combination;
  //  value is also an integer.
 public:
     SparseMatrix ( int r, int c, int t);
     // creates a r×c SparseMatrix with a capacity of t nonzero
     // terms
     SparseMatrix Transpose ( );
          //  return the SparseMatrix obtained by transposing *this
     SparseMatrix  Add ( SparseMatrix b);
     SparseMatrix   Multiply ( SparseMatrix b);
 };
```

Sparse Matrix Representation

Use triple <row, col, value>, sorted in ascending order by <row, col>.

```
class  SparseMatrix;
class  MatrixTerm {
friend class SparseMatrix;
Private:
   int row, col, value;
};
```

We need also

the number of rows

the number of columns

the number of nonzero elements

And in class SparseMatrix:

private:

```
Int rows, cols, terms, capacity;
MatrixTerm *smArray;
```

Now we can store the matrix of Fig.2.2 (b) as Fig.2.3 (a).

	row	col	value
smArray[0]	0	0	15
[1]	0	3	22
[2]	0	5	-15
[3]	1	1	11
[4]	1	2	3
[5]	2	3	-6
[6]	4	0	91
[7]	5	2	28

Fig.2.3 (a)

Transposing a Matrix

Transpose:

If an element is at position [i][j] in the original matrix, then it is at position [j][i] in the transposed matrix.

Fig.2.3(b) shows the transpose of Fig2.3(a).

```
for(col=0;col<n;col++)
    for(row=0;row<m;row++)
        n[col][row]=m[row][col];
T(n)=O(m×n)
```

M(a)

	i	j	v
0	6	7	8
1	1	2	12
2	1	3	9
3	3	1	-3
4	3	6	14
5	4	3	24
6	5	2	18
7	6	1	15
8	6	4	-7

M(b)

	i	j	v
0	7	6	8
1	1	3	-3
2	1	6	15
3	2	1	12
4	2	5	18
5	3	1	9
6	3	4	24
7	4	6	-7
8	6	3	14

?

First try:

For (each row i)

✓take element (i, j, value)

✓store it in (j, i, value) of the transpose;

Difficulty:

NOT knowing where to put (j, i, value) until all other elements preceding it have been processed.

smArray	row	col	value
✓[0]	0	0	15
[1]	0	4	91
[2]	1	1	11
[3]	2	1	3
[4]	2	5	28
[5]	3	0	22
[6]	3	2	-6
[7]	5	0	-15

Improvement:

For (all elements in col j)

✓store (i, j, value) of the original matrix as

✓ (j, i, value) of the transpose;

Since the rows are in order, we will locate elements in the correct column order.

smArray	row	col	value
✓[0]	0	0	15
[1]	0	4	91
[2]	1	1	11
[3]	2	1	3
[4]	2	5	28
[5]	3	0	22
[6]	3	2	-6
[7]	5	0	-15

ptr		i	j	v	ptr
	0	6	7	8	
p→	1	1	2	12	←p
p→	2	1	3	9	←p
p→	3	3	1	-3	←p
p→	4	3	6	14	←p
p→	5	4	3	24	←p
p→	6	5	2	18	←p
p→	7	6	1	15	←p
p→	8	6	4	-7	←p

ma

ptr		i	j	v
	0	7	6	8
k→	1	1	3	-3
k→	2	1	6	15
k→	3	2	1	12
k→	4	2	5	18
k→	5	3	1	9
	6	3	4	24
	7	4	6	-7
	8	6	3	14

mb

col=1 col=2

```
SparseMatrix SparseMatrix::Transpose ( )
{ // return the transpose of *this
    SparseMatrix b(cols, rows, terms);
    if (terms > 0)
    {   //nonzero matrix
      int currentB = 0;
```

```
      for ( int c=0; c<cols; c++ ) // transpose by columns
        for ( int i=0; i<terms; i++ )
        // find and move terms in column c
          if ( smArray[i].col == c )
          {
            b.smArray[CurrentB].row = c;
            b.smArray[CurrentB].col = smArray[i].row;
            b.smArray[CurrentB++].value= smArray[i].value;
          }
    } // end of if (terms > 0)
  return b;
}
```

Time complexity of Transpose:

- **line 7-15 loop--- cols times**
- **line 10 loop--- terms times**
- **other line--- O(1)**

Total time: O(cols* terms)

Additional space: O(1)

Think:

O(cols* terms) is not good. If terms = O(cols* rows) then it becomes O($cols^2$* rows)---too bad!

Since with 2-dimensional representation, we can get an easy O(cols* rows) algorithm as:

```
for (int j=0;j < columns;j++)
    for (int i=0; i < rows; i++)  B[j][i] = A[i][j];
```

Further improvement:

If we use some more space to store *some knowledge* about the matrix, we can do much better: doing it in O(cols + terms).

ma

	i	j	v
0	6	7	8
1	1	2	12
2	1	3	9
3	3	1	-3
4	3	6	14
5	4	3	24
6	5	2	18
7	6	1	15
8	6	4	-7

?

mb

	i	j	v
0	7	6	8
1	1	3	-3
2	1	6	15
3	2	1	12
4	2	5	18
5	3	1	9
6	3	4	24
7	4	6	-7
8	6	3	14

- **get the number of elements in each column of *this = the number of elements in each row of B;**
- **obtain the starting point in B of each of its rows;**
- **move the elements of *this one by one into their right position in B.**

Now the algorithm FastTranspose.

col	1	2	3	4	5	6	7
num[col]	2	2	2	1	0	1	0
cpot[col]	1	3	5	7	8	8	9
	2	4	6			9	
	3	5	7				

	i	j	v
0	6	7	8
1	1	2	12
2	1	3	9
3	3	1	-3
4	3	6	14
5	4	3	24
6	5	2	18
7	6	1	15
8	6	4	-7

ma

	i	j	v
0	7	6	8
1	1	3	-3
2	1	6	15
3	2	1	12
4	2	5	18
5	3	1	9
6	3	4	24
7	4	6	-7
8	6	3	14

mb

```
SparseMatrix SparseMatrix::FastTranspos ( )
{ // return the transpose of *this in O(terms+cols) time.
   SparseMatrix b(cols, rows, terms);
   if (terms > 0)
   { // nonzero matrix
     int *rowSize = new int[cols];
     int *rowStart = new int[cols];
     // compute rowSize[i] = number of terms in row i of b
     fill(rowSize, rowSize + cols, 0);  // initialze
     for (i=0; i<terms; i++ )  rowSize[smArray[i].col]++;
```

```
// rowStart[i] = starting position of row i in b
rowStart[0] = 0;
for (i=1;i<cols;i++) rowStart[i]=rowStart[i-1]+rowSize[i-1];
for (i=0; i<terms; i++)
{     // copy from *this to b
      int j = rowStart[smArray[i].col];
      b.smArray[j].row = smArray[i].col;
      b.smArray[j].col = smArray[i].row;
      b.smArray[j].value = smArray[i].value;
      rowStart[smArray[i].col]++;
 }   // end of for
```

```
22      delete [ ] rowSize;
23      delete [ ] rowStart;
24  }  // end of if
25  return b;
26 }
```

After line 13, we get :

	[0]	[1]	[2]	[3]	[4]	[5]
RowSize=	2	1	2	2	0	1
RowStart=	0	2	3	5	7	7

Note the error in P101 of the text book!

Analysis:

3 loops:

- **line 10--- O(terms)**
- **line 13--- O(cols)**
- **line 14 – 21--- O(terms)**

and line 9--- O(cols), other lines--- O(1)

Total: O(cols+terms)

This is a typical example for trading space for time.

Exercises: P107-1, 2, 4

The String Abstract data Type

A string $S = s_0, s_1, \ldots, s_{n-1}$,

where $s_i \in$ char, $0 \le i < n$, n is the length.

ADT 2.5 String

class String

{ // a finite set of zero or more characters;

public:

String (**char** *init, **int** m);

// initialize ***this** to string init of length m

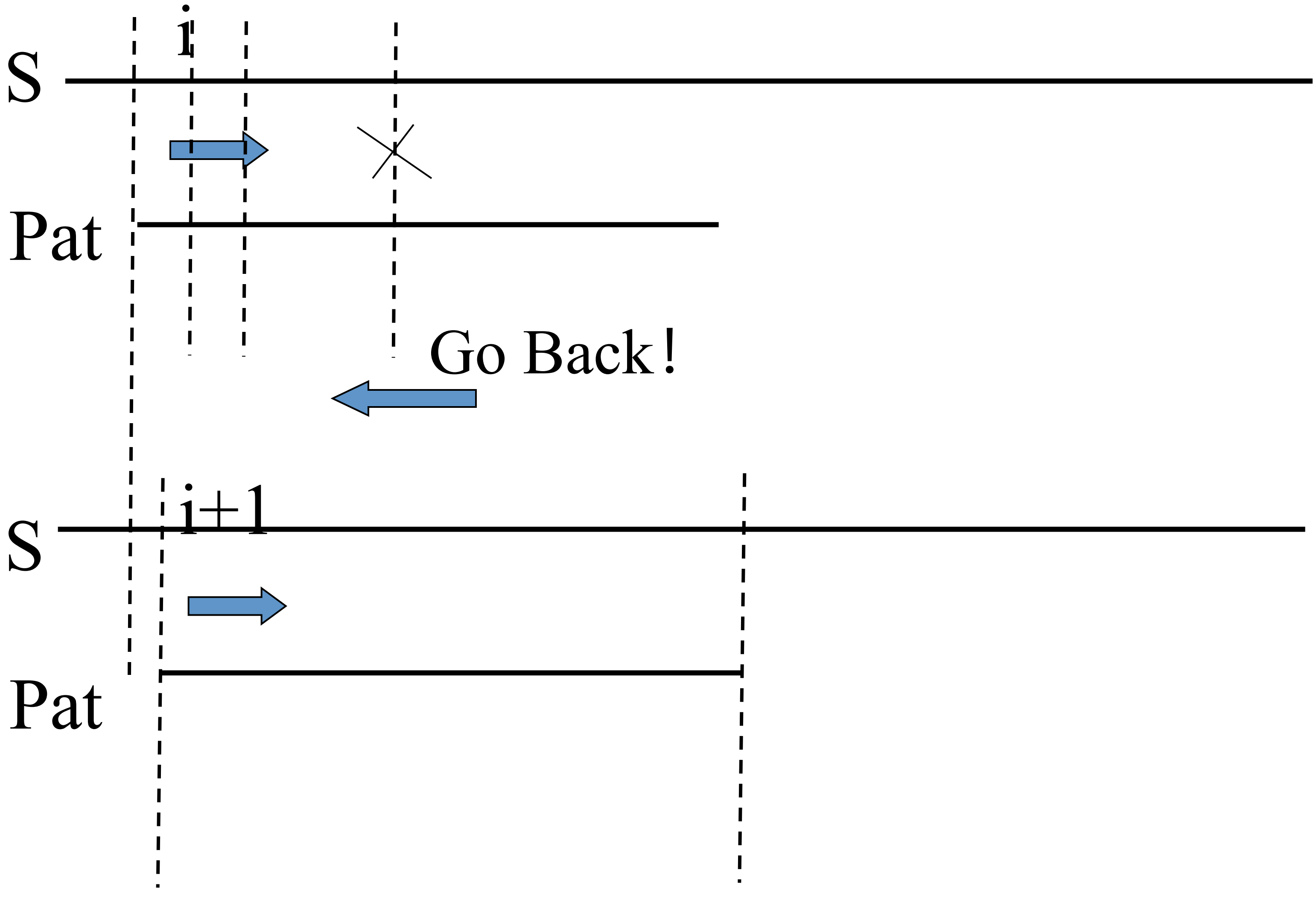
```
bool operator == (String t );
// if *this equals t, return true else false.
bool operator ! (  );
// if *this is empty return true else false.
int Length ( );
 // return the number of chars in *this
 String Concat (String t);
 String Substr (int  i, int  j);
 int Find (String pat);
// return i such that pat matches the substring of *this that
// begins at position i. Return –1 if pat is either empty or not
// a substring of *this.
};
```

Assume the String class is represented by:

private:

char* str**;**

String Pattern Matching: A Simple Algorithm



```
int String::Find ( String pat )
{ // Return –1 if pat does not occur in *this; otherwise
  // return the first position in *this, where  pat begins.
    if (pat.Length( ) == 0) return -1; // pat is empty
    for (int start=0; start<=Length( ) - pat.Length( ); start++)
    { //  check  for match beginning at str[start]
        for (int j=0; j<pat.Length( )&&str[start+j]==pat.str[j];j++)
            if (j== pat.Length( )) return start; // match found
        // no match at position start
          }
     return -1;     // pat does not occur in s
}
```

The complexity of it is O(LengthP * LengthS).

Problem:

rescanning.

Even if we check the last character of pat first, the time complexity can’t be improved!

String Pattern Matching: The Knuth-Morris-Pratt Algorithm

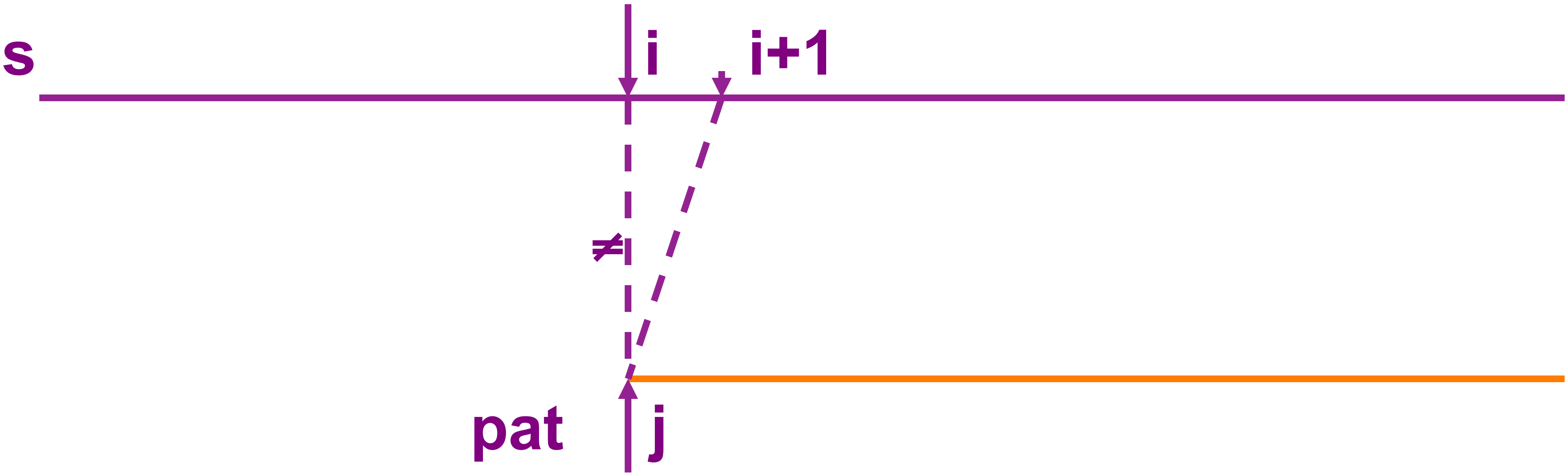
Can we get an algorithm which *avoid rescanning* the strings and works in O(LengthP + LengthS)?

This is optimal for this problem, as in the worst it is necessary to look at characters in the pattern and string at least once.

Basic Ideas:

- **Rescanning to avoid missing the target ---**
  - **too conservative**
- **If we can go without rescanning, it is likely to do the job in O(LengthP + LengthS).**
- **Preprocess the pattern, to get some knowledge of the characters in it and the position in it, so that if a mismatch occurs we can determine where to continue the search and avoid moving backwards in the string.**

Now we show details about the idea.



case: $j = 0$

An concrete example:

i

↓

s = . . . a b d a b ? . . . .

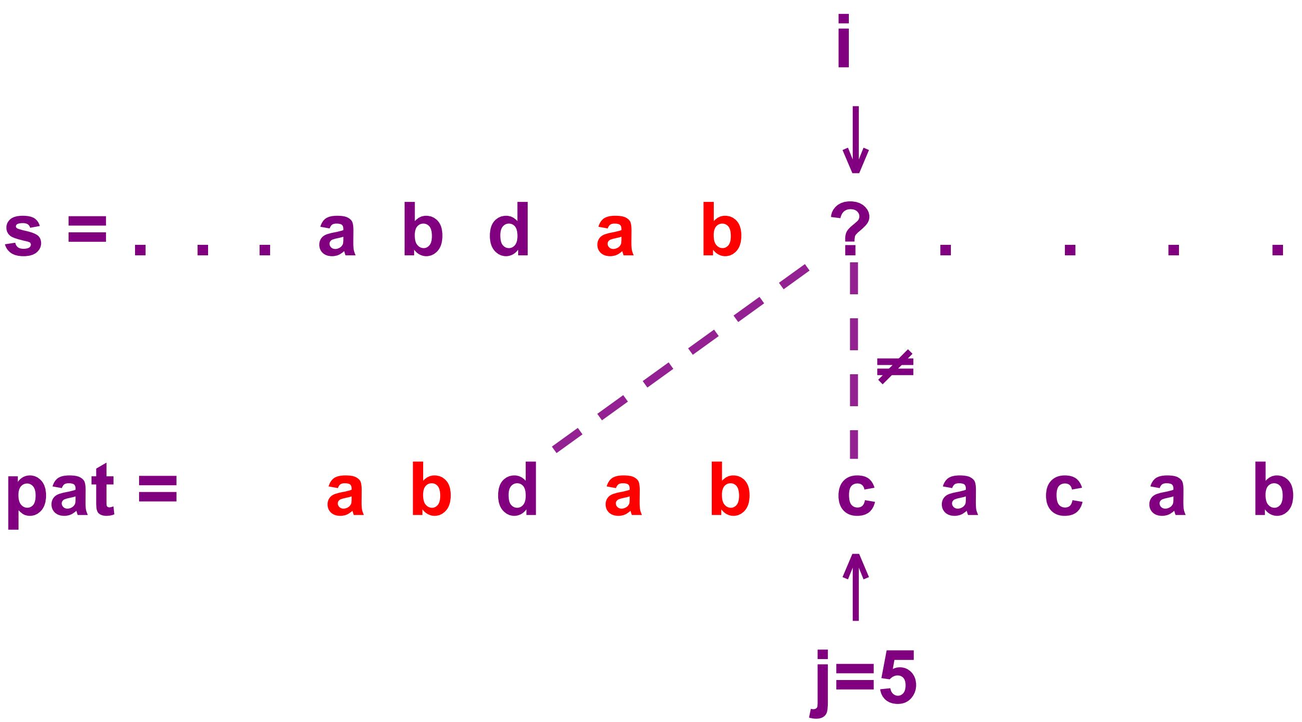
≠

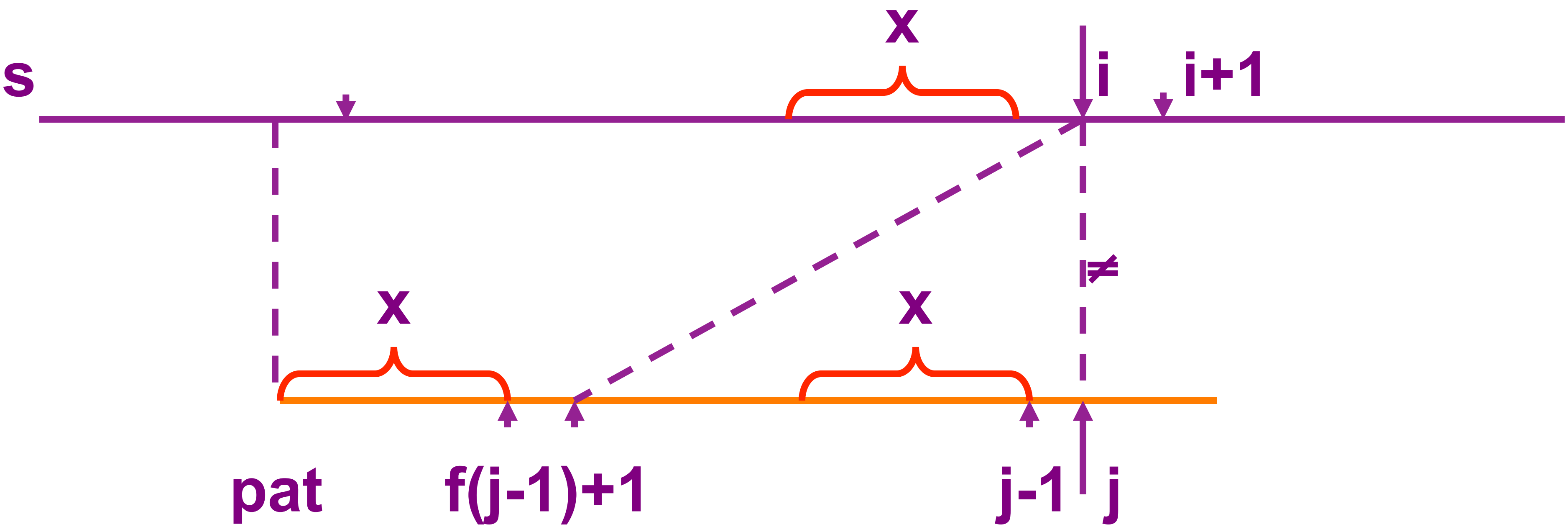
pat = a b d a b c a c a b

↑

j=5

An concrete example:





case: $j \neq 0$

To formalize the above idea:

Definition: if $p=p_0p_1\ldots p_{n-1}$ is a pattern, then its failure function f, is defined as:

$$f(j) = \begin{cases} \text{largest } k < j \text{, such that } p_0p_1\ldots p_k = p_{j-k}p_{j-k+1}\ldots p_j & \text{if such } k \geq 0 \text{ exists} \\ -1 & \text{otherwise} \end{cases}$$

For example, pat = a b c a b c a c a b, we have

j	0	1	2	3	4	5	6	7	8	9
pat	a	b	c	a	b	c	a	c	a	b
f	-1	-1	-1	0	1	2	3	-1	0	1

Note:

- largest : no match be missed
- k < j : avoid dead loop

From the definition of f, we have the following rule for pattern matching:

If a partial match is found such that $s_{i-j}\ldots s_{i-1} = p_0 p_1 \ldots p_{j-1}$ and $s_i \neq p_j$ then matching may be resumed by comparing s_i and $p_{f(j-1)+1}$ if $j \neq 0$.

If $j=0$, then we may continue by comparing s_{i+1} and p_0.

The failure function is represented by an array of integers f, which is a private data member of String.

Now the algorithm FastFind.

```
int String::FastFind (String pat)
{ // Determine if pat is a substring of s
   int PosP = 0,  PosS = 0;
   int LengthP= pat.Length( ), LengthS= Length( );
   while ((PosP  < LengthP) && (PosS  < LengthS))
      if ( pat.str[PosP] == str[PosS] ) { // characters match
      PosP ++; PosS ++;
      }
      else
        if ( PosP==0)
           PosS++;
        else PosP= pat.f [PosP-1] + 1;
   if ((PosP < LengthP) || LengthP==0)) return –1;
    else return PosS  - LengthP ;
}
```

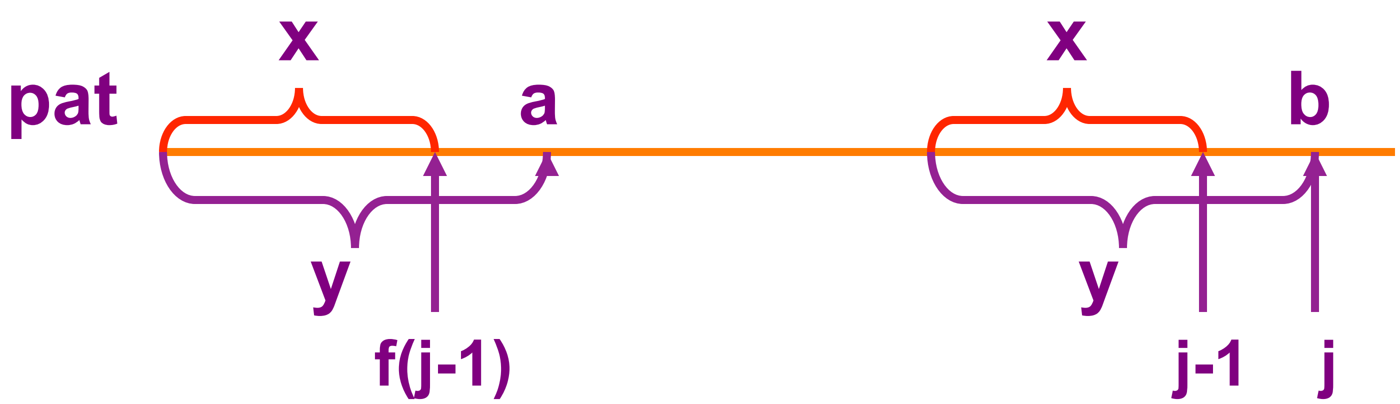
Analysis of FastFind:

- **Line 7 and 11 --- at most LengthS times, since PosS is increased but never decreased. So PosP can move right on pat at most LengthS times (line 7).**
- **Line 12 moves PosP left, it can be done at most LengthS times. Note that f(j-1)+1< j.**

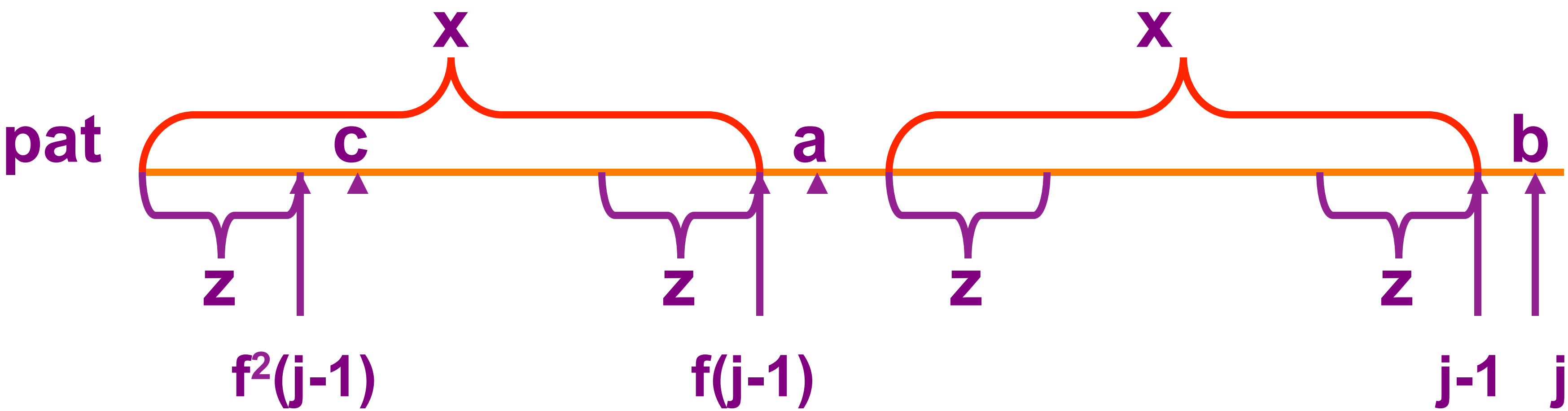
Consequently, the computing time is O(LengthS).

How about the computing of the f for the pattern? By similar idea, we can do it in O(LengthP).

f(0)=-1, now if we have f(j-1), we can compute f(j) from it by the following observation:



If a=b, then f(j)=f(j-1)+1 else



If c=b, $f(j)=f(f(j-1))+1=f^2(j-1)+1$ else ……

In general, we have the following restatement of the failure function:

$$f(j) = \begin{cases} -1 & \text{if } j=0 \\ f^m(j-1)+1 & \text{where m is the least k for which } p_{f^k(j-1)+1} = p_j \\ -1 & \text{if there is no k satisfying the above} \end{cases}$$

Now we get the algorithm to compute f.

```
void String::Failurefunction( )
{ // compute the failure function of the pattern *this.
   int LengthP= Length( );
   f [0]= -1;
   for (int j=1; j< LengthP; j++) // compute  f[j]
     {
      int i=f [j-1];
      while ((*(str+j)!=*(str+i+1)) && (i>=0)) i=f[i]; // try for m
       if ( *(str+j)==*(str+i+1))
           f[j]=i+1;
            else f[j]= -1;
   }
}
```

Analysis of fail:

- **In each iteration of the while i decreases (line 8, and f(j)<j)**
- **i is reset (line 7) to –1 (when the previous iteration went through line 11), or to a value 1 greater than its value on the previous iteration (when through line 10).**
- **There are only LengthP –1 executions of line 7, the value of i has a total increment of at most LengthP –1.**
- **i cannot be decremented more than LengthP –1 times, the while is iterated at most LengthP –1 times over the whole algorithm.**

Consequently, the computing time is O(LengthP).

Now we can see, when the failure function is not known in advance, pattern matching can be carried out in O(LengthP + LengthS) by first computing the failure function and then using the FastFind.

Exercises: P118-1, P119-7, 9

Experiment 1: P123-8