最优化方法

东南大学

计算机&人工智能学院

宋沫飞

songmf@seu.edu.cn

对偶

- **Lagrange**对偶问题
- 弱对偶和强对偶
- 几何解释
- 最优性条件
- 扰动和灵敏度分析
- 例子
- 广义不等式

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$$= \begin{cases} -\mathbf{1}^T \nu & W + \mathbf{diag}(\nu) \succeq 0 \\ -\infty & \text{otherwise} \end{cases}$$

- 下界的性质：

$$p^\star \geq -\mathbf{1}^T \nu \text{ if } W + \mathbf{diag}(\nu) \succeq 0$$

- 例如：

双向划分问题

$$\begin{array}{ll}\text{minimize} & x^T W x \\ \text{subject to} & x_i^2 = 1, \quad i = 1, \ldots, n\end{array}$$

- 非凸优化问题，可行集包含2^n个离散点
- 对偶函数

$$\begin{aligned} g(\nu) = \inf_x (x^T W x + \sum_i \nu_i (x_i^2 - 1)) &= \inf_x x^T (W + \mathbf{diag}(\nu)) x - \mathbf{1}^T \nu \\ &= \begin{cases} -\mathbf{1}^T \nu & W + \mathbf{diag}(\nu) \succeq 0 \\ -\infty & \text{otherwise} \end{cases} \end{aligned}$$

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- 例如：
- $$\nu = -\lambda_{\min}(W) \mathbf{1} \qquad p^\star \geq n \lambda_{\min}(W)$$

Lagrange对偶和共轭函数

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- 共轭函数:

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$$\begin{aligned} g(\lambda, \nu) &= \inf_{x \in \mathbf{dom}\, f_0} \left(f_0(x) + (A^T\lambda + C^T\nu)^T x - b^T\lambda - d^T\nu \right) \\ &= -f_0^*(-A^T\lambda - C^T\nu) - b^T\lambda - d^T\nu \end{aligned}$$

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- 如：熵的最大化

$$f_0(x) = \sum_{i=1}^n x_i \log x_i, \qquad f_0^*(y) = \sum_{i=1}^n e^{y_i - 1}$$

对偶问题

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g(\lambda, \nu) = \inf_x L(x, \lambda, \nu) = \begin{cases} -b^T\nu & A^T\nu - \lambda + c = 0 \\ -\infty & \text{otherwise} \end{cases}
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- D的相对内部：
  - $\text{Rel int}\, D = \{x \in D \mid B(x,r) \cap \text{aff}\, D \subset D, \exists r > 0\}$

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$$\exists x \in \textbf{relint}\, \mathcal{D} \qquad f_i(x) < 0, \quad i = 1, \ldots, m, \qquad Ax = b$$

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  - 当该问题是严格可行的，即

$$\exists x \in \textbf{relint}\, \mathcal{D} \qquad f_i(x) < 0, \quad i = 1, \dots, m, \qquad Ax = b$$

  - 则为强对偶，即满足 $d^\star = p^\star$
- 弱化：
  - 若不等式约束为仿射时，只要可行域非空，必有

$$d^\star = p^\star$$

线性规划的不等式形式

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- 原问题

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$$g(\lambda) = \inf_x \left((c + A^T\lambda)^T x - b^T\lambda \right) = \begin{cases} -b^T\lambda & A^T\lambda + c = 0 \\ -\infty & \text{otherwise} \end{cases}$$

线性规划的不等式形式

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  subject to $Ax \preceq b$
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- 对偶问题

$$\begin{array}{ll} \text{maximize} & -b^T\lambda \\ \text{subject to} & A^T\lambda + c = 0, \quad \lambda \succeq 0 \end{array}$$

线性规划的不等式形式

- 原问题 $$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax \preceq b \end{array}$$
- 对偶函数

$$g(\lambda) = \inf_x \left((c + A^T\lambda)^T x - b^T\lambda\right) = \begin{cases} -b^T\lambda & A^T\lambda + c = 0 \\ -\infty & \text{otherwise} \end{cases}$$

- 对偶问题 $$\begin{array}{ll} \text{maximize} & -b^T\lambda \\ \text{subject to} & A^T\lambda + c = 0, \quad \lambda \succeq 0 \end{array}$$
- 根据**Slater**条件：若对某些$\tilde{x}$有$A\tilde{x} \prec b$，则$p^\star = d^\star$

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  - 必有 $p^\star = d^\star$

线性方程组的最小二乘解

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\begin{array}{ll}
\text{minimize} & x^T x \\
\text{subject to} & Ax = b
\end{array}
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- 对偶问题：

线性方程组的最小二乘解

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- 对偶问题：

$$\text{maximize} \quad -(1/4)\nu^T A A^T \nu - b^T \nu$$

线性方程组的最小二乘解

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- 强对偶成立

QCQP二次约束二次规划

QCQP二次约束二次规划

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\begin{array}{ll}
\text{minimize} & (1/2)x^T P_0 x + q_0^T x + r_0 \\
\text{subject to} & (1/2)x^T P_i x + q_i^T x + r_i \leq 0, \quad i = 1, \ldots, m,
\end{array}
$$

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- **Lagrangian:**

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- **Lagrangian:**
  - $L(x, \lambda) = \frac{1}{2} x^T P_0 x + q_0^T x + r_0 + \sum_{i=1}^{m} (\frac{1}{2} \lambda_i x^T P_i x + \lambda_i q_i^T x + \lambda_i r_i)$

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  - $= \frac{1}{2}x^T(P_0 + \sum_{i=1}^{m}\lambda_i P_i)x + (q_0 + \sum_{i=1}^{m}\lambda_i q_i)^T x + (r_0 + \sum_{i=1}^{m}\lambda_i r_i)$

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$\boxed{P(\lambda)}$ $\boxed{q(\lambda)}$ $\boxed{r(\lambda)}$

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- **存在*x*，满足**

$$(1/2)x^T P_i x + q_i^T x + r_i < 0, \quad i = 1, \ldots, m$$

几何解释

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- 考虑只含一个约束 $f_1(x) \leq 0$的问题

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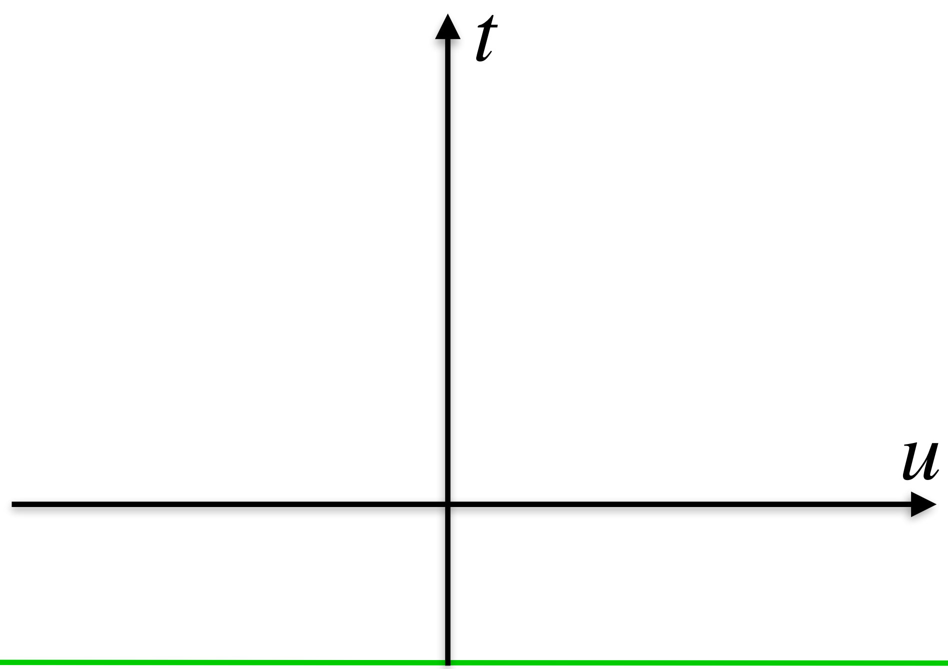

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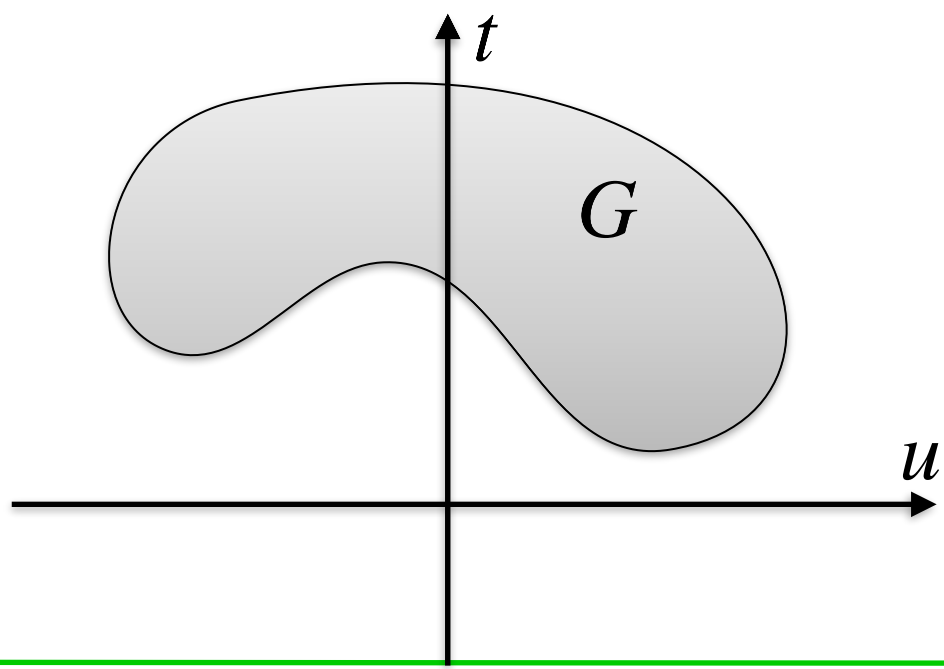

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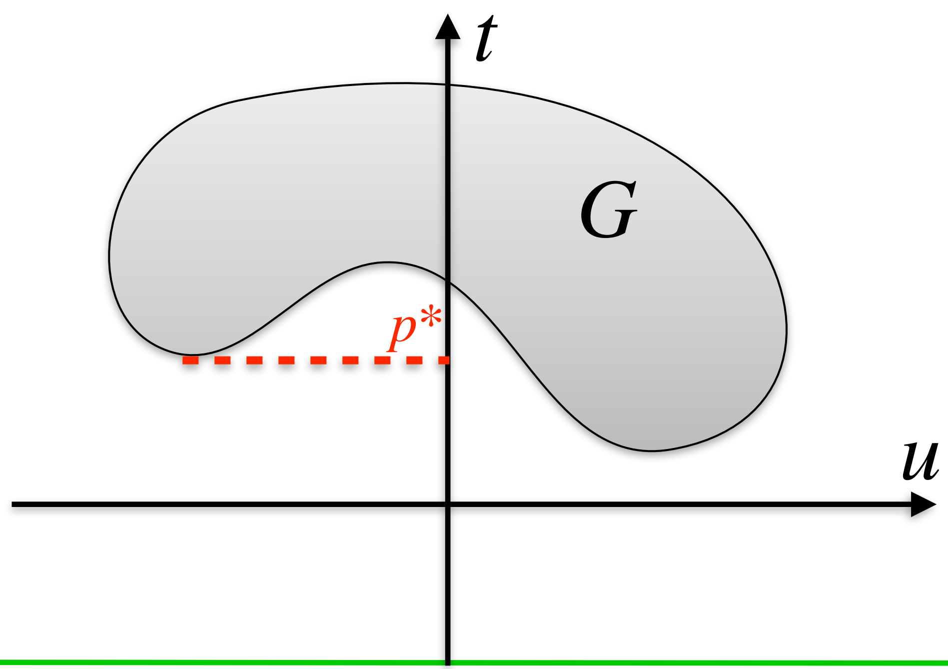

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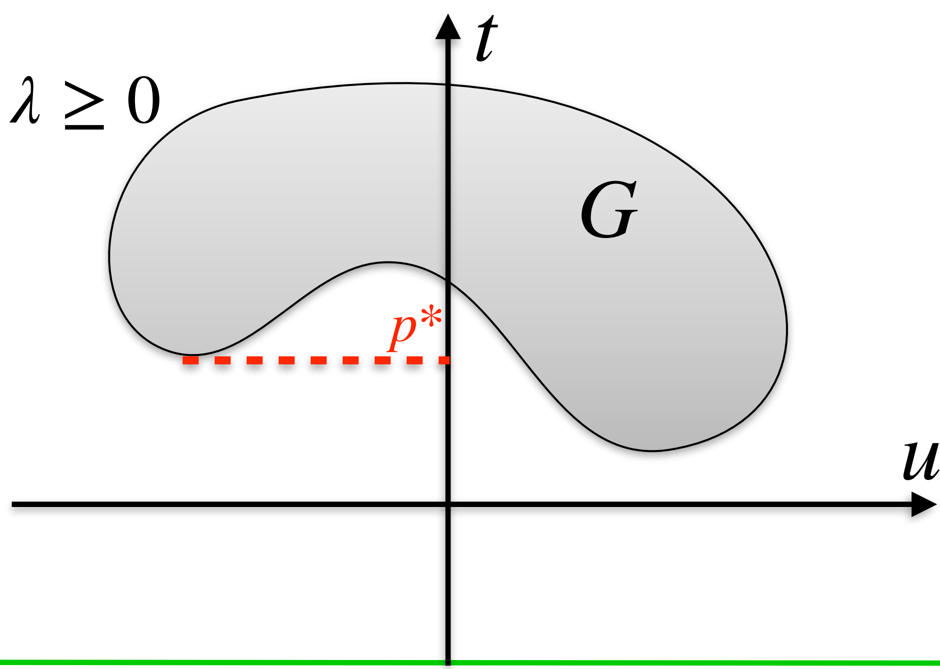
$g(\lambda)$ $\lambda \geq 0$



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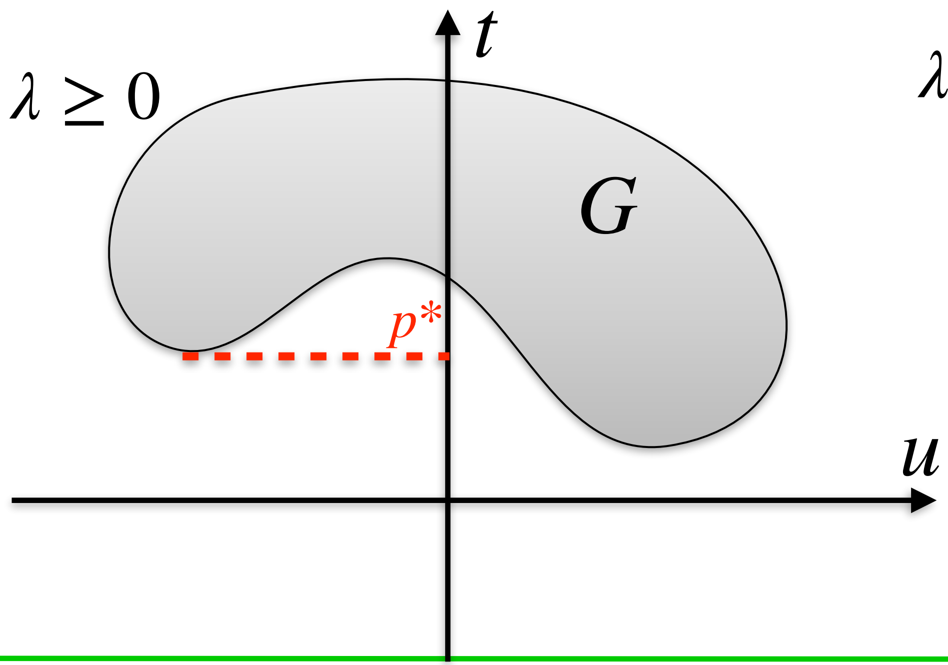
$g(\lambda)$ $\lambda \ge 0$

$\lambda u + t = 0$



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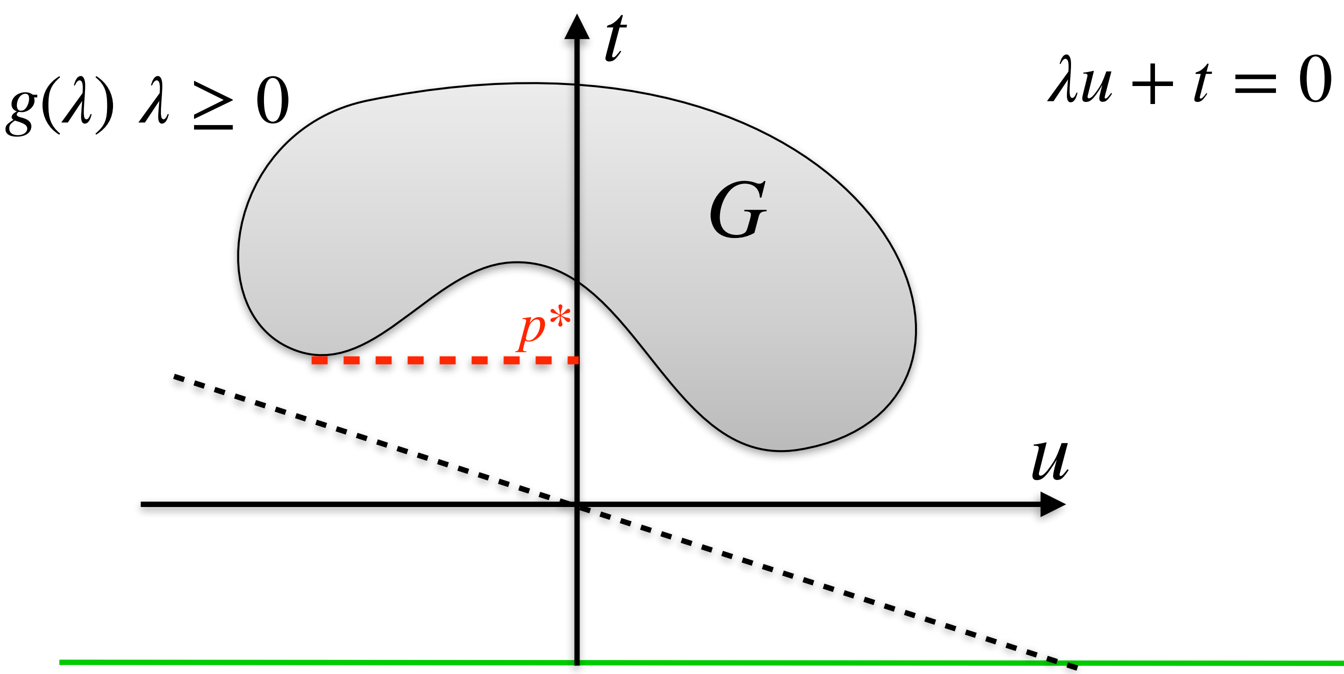

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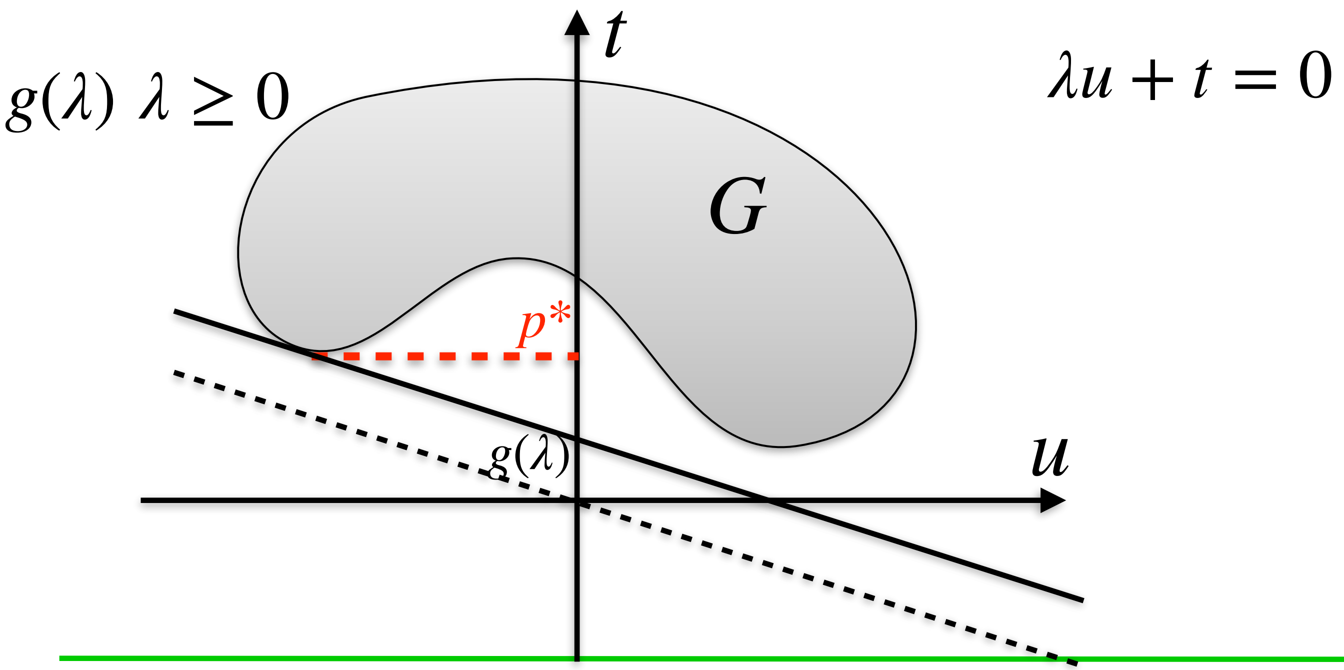

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$g(\lambda)\ \lambda \ge 0$

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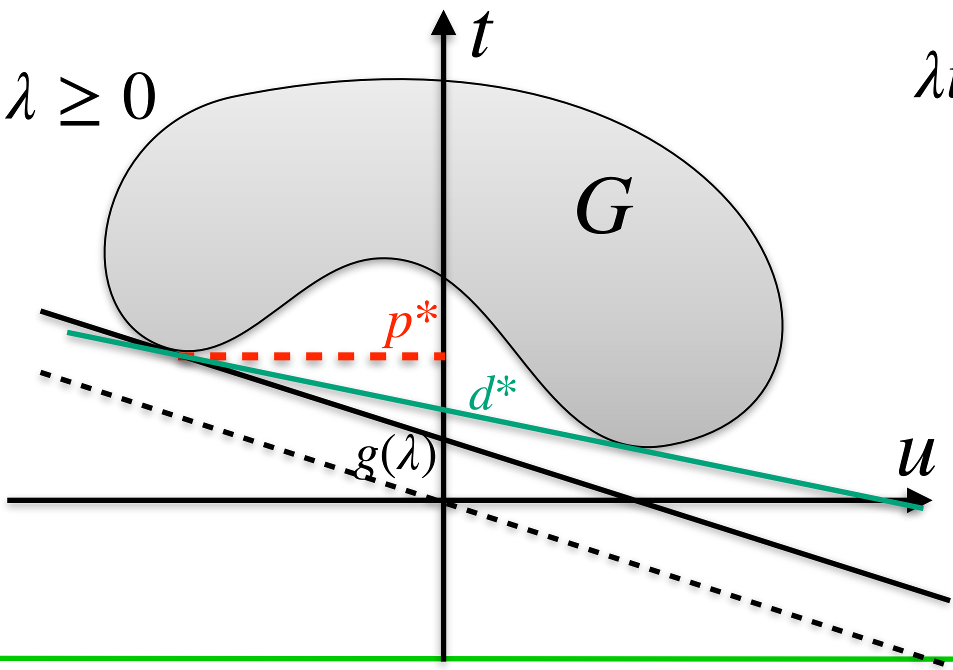

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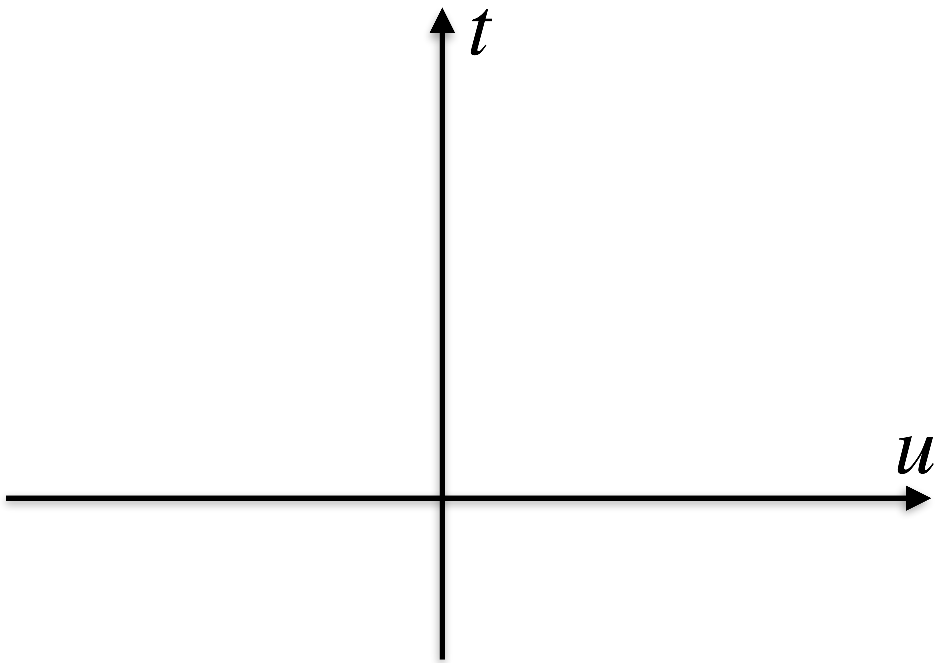

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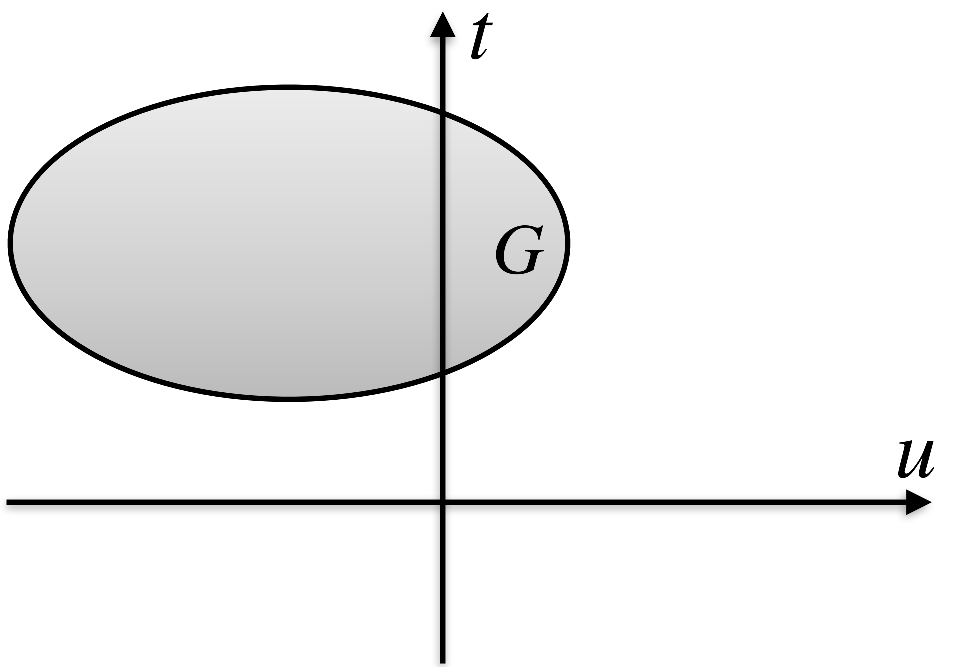

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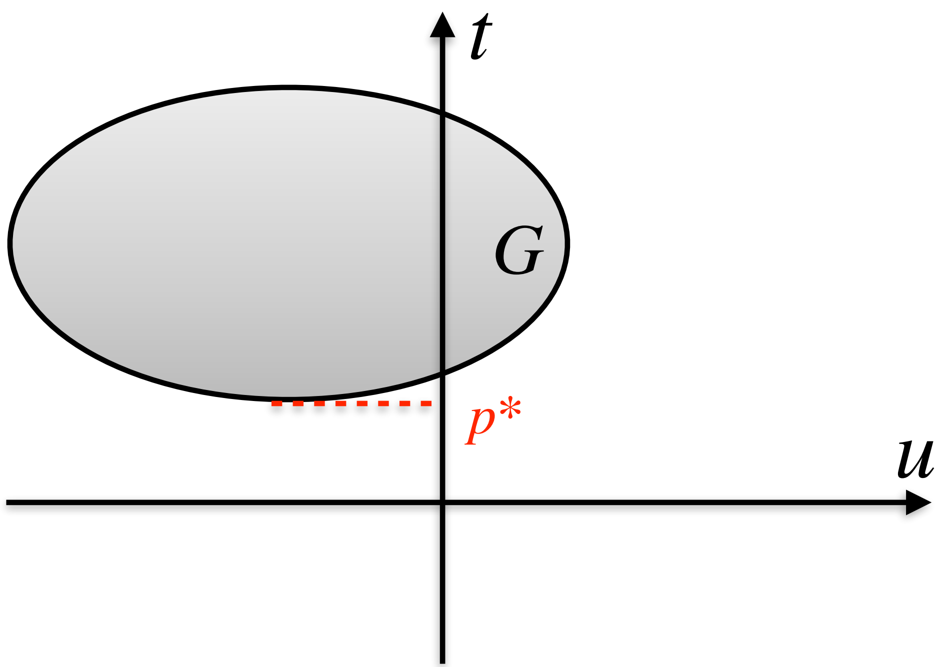

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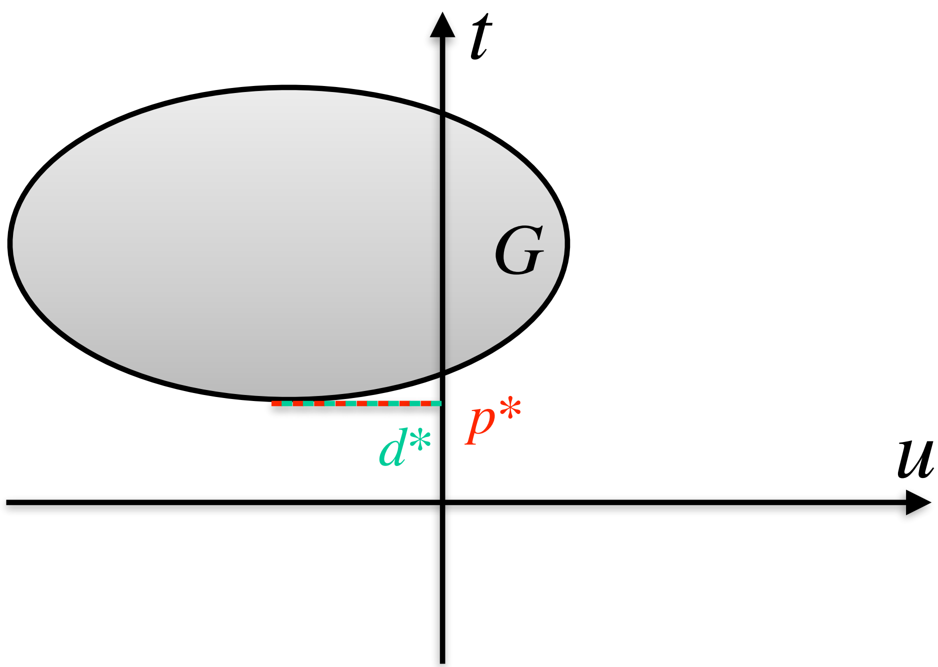

经济学解释

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$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & f_i(x) \leq 0, \quad i = 1, \ldots, m \end{array}$$

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- x：产品数量

经济学解释

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- x：产品数量
- $f_i(x) \le 0$：原材料

经济学解释

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & f_i(x) \le 0, \quad i = 1, \ldots, m \end{array}$$

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- $f_i(x) \le 0$：原材料
- $-f_0(x)$：利润

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❑ 假设原材料可买可卖，第i种原材料价格为$\lambda_i(\lambda_i \ge 0)$

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  - $d* = \max\limits_{\lambda \ge 0} g(\lambda) \le p*$

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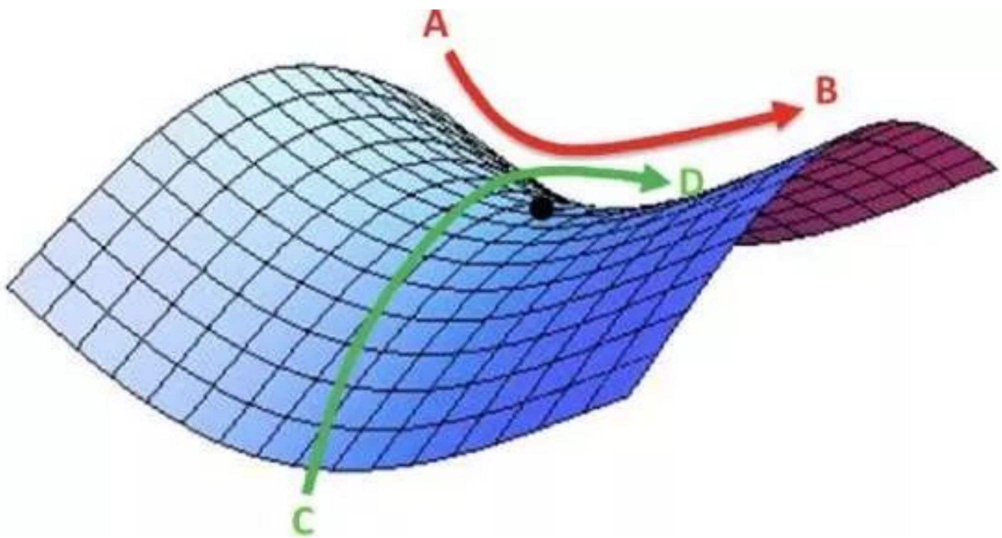

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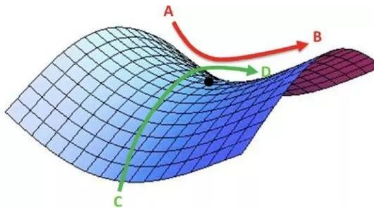

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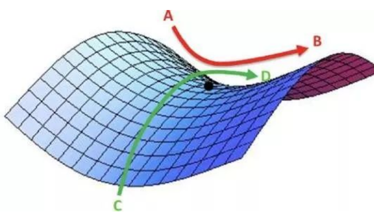

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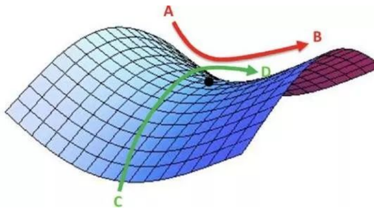

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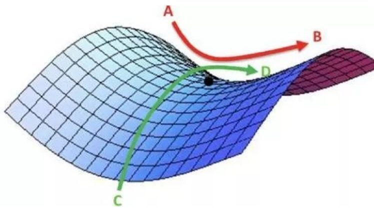

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- $f_0(\tilde{x}) = f_0(\tilde{x}) + \sum_{i=1}^{m} \tilde{\lambda}_i f_i(\tilde{x}) = L(\tilde{x}, \tilde{\lambda})$
- $f_0(\tilde{x}) = \inf_x \{f_0(x) + \sum_{i=1}^{m} \tilde{\lambda}_i f_i(x)\}$ $\qquad f_i(\tilde{x}) \le 0,\ \tilde{\lambda} \ge 0$

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互补松弛性

互补松弛性

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & f_i(x) \leq 0, \quad i = 1, \ldots, m \\ & h_i(x) = 0, \quad i = 1, \ldots, p \end{array}$$

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互补松弛性

m

su

$$\sum_{i=1}^{m}\lambda_i^* f_i(x^*)+\sum_{i=1}^{p}v_i^* h_i(x^*)=0$$

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- 互补松弛

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m

su

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- 对

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KKT条件

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    $$\lambda_i f_i(x) = 0,\ i = 1, \ldots, m$$

  - **Lagrangian**梯度的定常方程式（稳定性条件）：

    $$\nabla f_0(x) + \sum_{i=1}^{m} \lambda_i \nabla f_i(x) + \sum_{i=1}^{p} \nu_i \nabla h_i(x) = 0$$

凸问题的KKT条件

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  - $\frac{\partial L(x, \tilde{\lambda}, \tilde{v})}{\partial x}\Big|_{x=\tilde{x}} = 0$，则$\tilde{x}$为极小值点

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  - $g(\tilde{\lambda}, \tilde{v}) = \inf_x L(x, \tilde{\lambda}, \tilde{v}) = L(\tilde{x}, \tilde{\lambda}, \tilde{v})$
  - $= f_0(\tilde{x}) + \sum_{i=1}^{m} \tilde{\lambda}_i f_i(\tilde{x}) + \sum_{i=1}^{p} \tilde{v}_i h_i(\tilde{x}) = f_0(\tilde{x})$

例

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- $$\min \frac{1}{2}x^T P x + q^T x + r, P \in S_+^n$$
  $$\textbf{s.t. } Ax = b$$

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    $$\Leftrightarrow Px^* + q + A^T v^* = 0$$
- $$\begin{bmatrix} P & A^T \\ A & 0 \end{bmatrix} \begin{bmatrix} x^* \\ v^* \end{bmatrix} = \begin{bmatrix} -q \\ b \end{bmatrix}$$

例：注水

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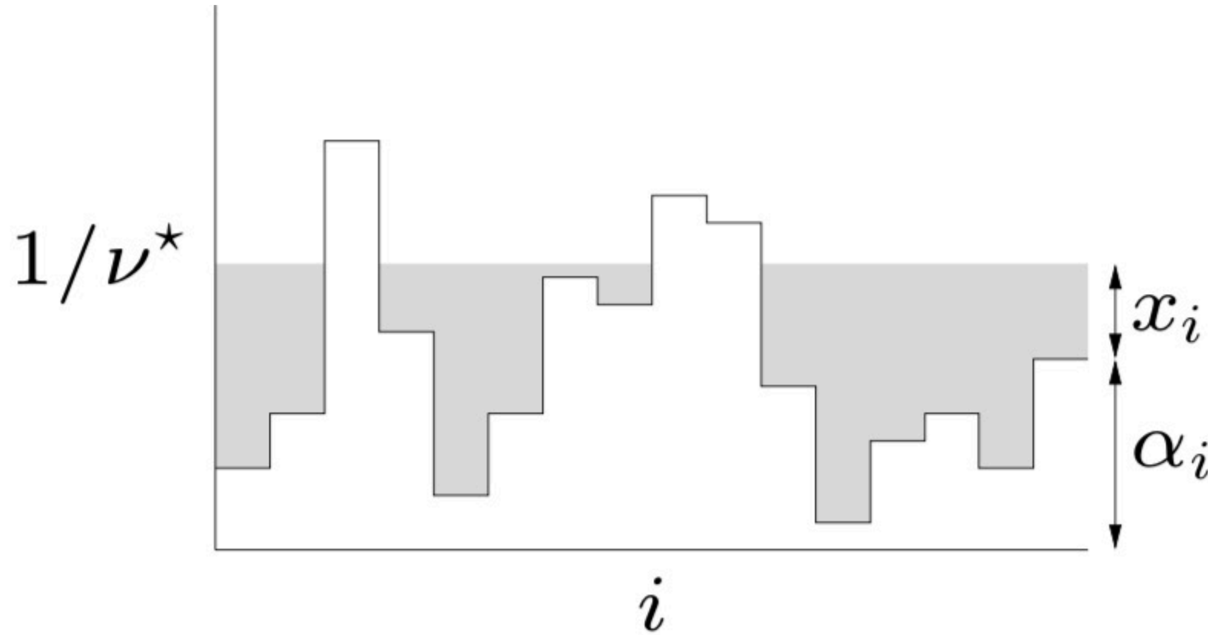
- $\begin{cases} v^* \ge 1/\alpha_i \to & x_i^* = 0 \\ v^* < 1/\alpha_i \to & x_i^* > 0 \\ & x_i^* = 1/v^* - \alpha \end{cases}$

例：注水



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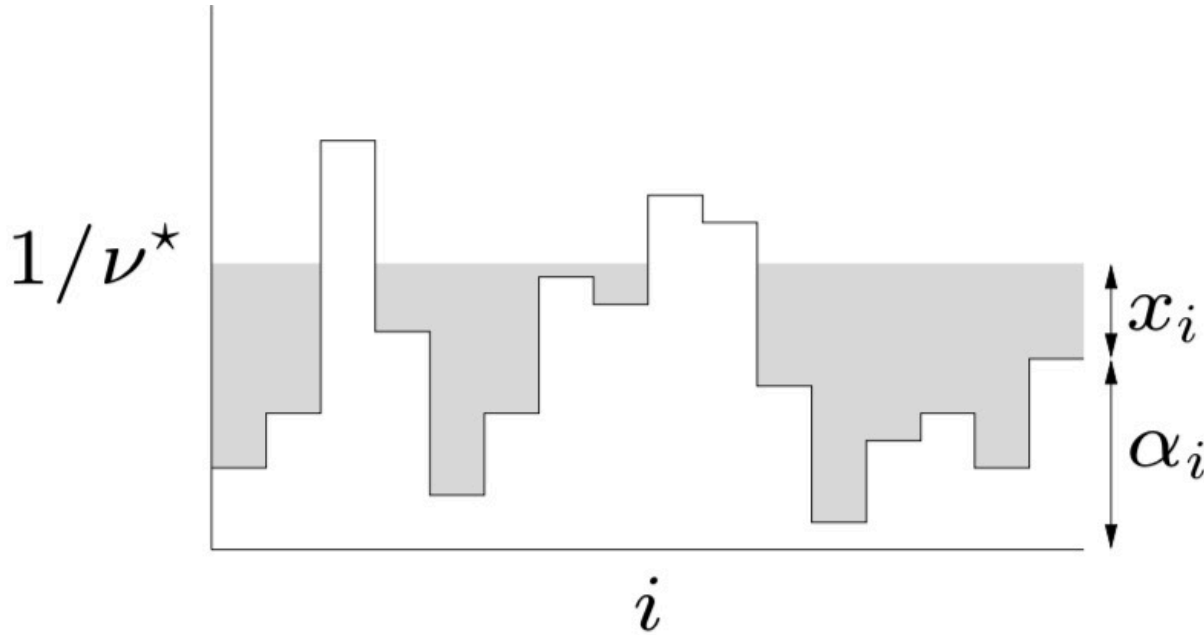

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- **KKT条件**

$$x^* \geq 0$$

$$\lambda^* \geq 0$$

$$\lambda_i^*(-x_i^*) = 0$$

$$\nabla f_0(x^*) + (-\lambda^*) = 0$$

对偶及问题重形式化

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  - 变换目标或约束函数变换

引入新变量和等式约束

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minimize $\quad f_0(Ax+b)$

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隐式约束

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- 带框约束的线性规划

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隐式约束

❑ 带框约束的线性规划

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❑ 将框约束隐式表达

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- 将框约束隐式表达

$$\begin{array}{ll} \text{minimize} & f_0(x) = \begin{cases} c^T x & -\mathbf{1} \preceq x \preceq \mathbf{1} \\ \infty & \text{otherwise} \end{cases} \\ \text{subject to} & Ax = b \end{array}$$

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扰动及灵敏度分析

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- 优化问题及其对偶问题

扰动及灵敏度分析

❑ 优化问题及其对偶问题

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & f_i(x) \le 0, \quad i = 1, \ldots, m \\ & h_i(x) = 0, \quad i = 1, \ldots, p \end{array}$$

$$\begin{array}{ll} \text{maximize} & g(\lambda, \nu) \\ \text{subject to} & \lambda \succeq 0 \end{array}$$

扰动及灵敏度分析

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\end{array}
\qquad
\begin{array}{ll}
\text{maximize} & g(\lambda, \nu) \\
\text{subject to} & \lambda \succeq 0
\end{array}
$$

- 扰动问题及其对偶

$$
\begin{array}{lll}
\text{min.} & f_0(x) & \\
\text{s.t.} & f_i(x) \le u_i, & i = 1, \ldots, m \\
& h_i(x) = v_i, & i = 1, \ldots, p
\end{array}
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\begin{array}{ll}
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扰动及灵敏度分析

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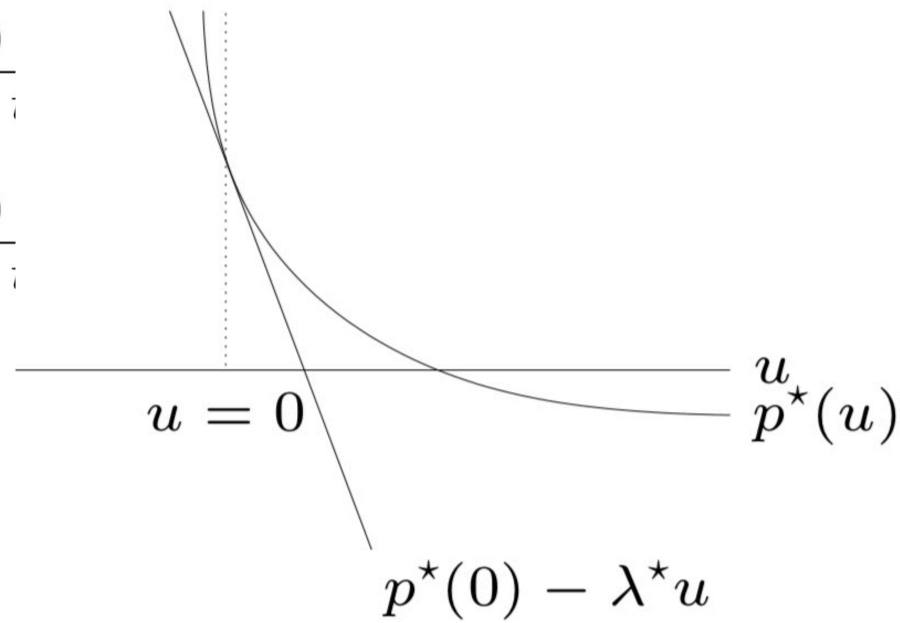

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&= g(\lambda_1, \ldots, \lambda_m, \nu)
\end{aligned}
$$

- 对所有可行解，$\tilde{x}$ 为最优值，有 $p^\star \geq g(\lambda_1, \ldots, \lambda_m, \nu)$

下界性质

- 若 $\lambda_i \succeq_{K_i^*} 0$， 则 $g(\lambda_1, \ldots, \lambda_m, \nu) \le p^\star$
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- 弱对偶： $p^\star \geq d^\star$ ，强对偶：$p^\star = d^\star$

半定规划

半定规划

- 原问题

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & x_1 F_1 + \cdots + x_n F_n \preceq G \end{array}$$

半定规划

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- **Lagrange**乘子为矩阵 $Z \in \mathbf{S}^k$

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$$L(x, Z) = c^T x + \mathbf{tr}\left(Z(x_1 F_1 + \cdots + x_n F_n - G)\right)$$

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- 对偶函数

$$g(Z) = \inf_x L(x, Z) = \begin{cases} -\mathbf{tr}(GZ) & \mathbf{tr}(F_i Z) + c_i = 0, \quad i = 1, \ldots, n \\ -\infty & \text{otherwise} \end{cases}$$

半定规划

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半定规划

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- 若半定规划是严格可行的，则 $p^\star = d^\star$